Algebra

1. Find solutions to $x + 5 + \frac{6}{x} = 0$.

Solution:

Let x be a non-zero real number. Suppose $x + 5 + \frac{6}{x} = 0$.
Let's multiply both sides by x: $x^2 + 5x + 6 = 0$.
Notice that the left-hand side equals $(x+2)(x+3)$.
Therefore, $x = -2$ or $x = -3$ and, reciprocally, we can check that they're indeed solutions of the equation.

2. a, b, c and d are nonnegative real numbers. If we have: $a^2 + b^2 = 1$, $c^2 + d^2 = 1$ and $ac - bd = 12$, find $ad + bc$.

Solution:

(1) $a^2c^2 + b^2d^2 - abcd = (ac - bd)^2 = \frac{1}{4}$
(2) $a^2d^2 + b^2c^2 + abcd = (ad + bc)^2$
Therefore, adding (1) and (2):
$a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 = (ad + bc)^2 + \frac{1}{4}$
Notice that $a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 = (a^2 + b^2)(c^2 + d^2) = 1 \cdot 1 = 1$.
Consequently, $(ad + bc)^2 + \frac{1}{4} = 1$, so $(ad + bc)^2 = \frac{3}{4}$
Finally, since $ad + bc \geq 0$, we have $ad + bc = \frac{\sqrt{3}}{2}$.

3. Two numbers sum to 1337 and have a product 9001. Find the two numbers.

Solution:

Let a and b be the two numbers, with $a \leq b$.
Then we know that a and b are the solutions of the equation
"$x^2 - (a+b)x + abx = 0$", with $a + b = 1337$ and $ab = 9001$.
Using the quadratic formula, we find that
$a = \frac{1337 - \sqrt{1751565}}{2}$ and
$b = \frac{1337 + \sqrt{1751565}}{2}$

4. Find the sum of all the real and nonreal roots of $x^{2001} + \left(\frac{1}{2} - x\right)^{2001}$

Solution:

Binomial expansion: $\left(\frac{1}{2} - x\right)^{2001} = \sum_{k=0}^{2001} \left(\binom{2001}{k}\right) \frac{1}{2^{2001-k}} (-1)^k x^k$

Here, the coefficient before x^{2001} equals -1, so that term will cancel with x^{2001}: the degree of this polynomial is actually 2000.
Using Vieta's formula: the sum of the roots (s) of this polynomial equals $-\frac{a_{1999}}{a_{2000}}$ if we note $a_0, \cdots, a_{2000}$ the coefficients of this polynomial.
$a_{1999} = \left(\binom{2001}{1999}\right) \frac{1}{2^{2001-1999}}(-1)^{1999} = -250 \cdot 2001$
$a_{2000} = \left(\binom{2001}{2000}\right) \frac{1}{2^{2001-2000}}(-1)^{2000} = \frac{2001}{2}$
Therefore, $s = 500$.

5. Express $(0^3 - 350)(1^3 - 349)(2^3 - 348)\cdots(350^3 - 0)$ as concisely as possible.

Solution:

Well, $7^3 = 343$ so the product is equal to 0...
You can also notice it because the product is $\prod_{k=0}^{350}(k^3 + k - 350)$ and 7 is a root of the polynomial $x^3 + x - 350$.
For $k = 7$, we have $k^3 + k - 350 = 0$.

6. If $a + \frac{1}{a} = 3$, find $a^4 + \frac{1}{a^4}$.

Solution:

$\left(a + \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} + 2 = 3^2 = 9$ so $a^2 + \frac{1}{a^2} = 7$.
$\left(a^2 + \frac{1}{a^2}\right)^2 = a^4 + \frac{1}{a^4} + 2 = 7^2 = 49$ so $a^4 + \frac{1}{a^4} = 47$.

7. Let r and s be roots of $x^2 - 5x + 2$. Find $\frac{r^3-1}{r-1} + \frac{s^3-1}{s-1}$.

Solution:

According to Vieta's formulas, $r + s = -5$ and $rs = 2$.
Notice that $r \neq 1$ and $s \neq 1$ (because 1 isn't a root of this polynomial), so the expression is well defined.
Moreover, $\frac{r^3-1}{r-1} + \frac{s^3-1}{s-1} = r^2 + r + 1 + s^2 + s + 1$ because of the identity "$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$"
Therefore, $\frac{r^3-1}{r-1} + \frac{s^3-1}{s-1} = (r^2 + s^2) + (r + s) + 2$.
Notice that $r^2 + s^2 = (r+s)^2 - 2rs = 25 - 4 = 21$.
Therefore, $\frac{r^3-1}{r-1} + \frac{s^3-1}{s-1} = 21 - 5 + 2 = 18$.

8. f is a second degree polynomial such that $x^2 - 8x + 17 \leq f(x) \leq 2x^2 - 16x + 33$. If $f(10) = 55$, then find $f(22)$.

Solution:

Let $g(x) = x^2 - 8x + 17$ an $h(x) = 2x^2 - 16x + 33$ for all x.
Notice that $g(4) = h(4) = 1$, which is the minimum value of g and of h. Therefore $f(4) = 1$ and it's the minimum value of f.
Let's note $f(x) = ax^2 + bx + c$. We have the following equalities:

(1) $100a + 10b + c = 55$ because $f(10) = 55$
(2) $16a + 4b + c = 1$ because $f(4) = 1$
(3) $-\frac{b}{2a} = 4$ because the minimum value of f occurs at $x = 4$
(1) - (2) gives $84a + 6b = 54$, that is, $14a + b = 9$.
(3) gives $b = -8a$.
Therefore, $14a - 8a = 9$, so $a = \frac{3}{2}$. So we can deduce that $b = -12$ and $c = 25$.
Finally, $f(x) = \frac{3}{2}x^2 - 12x + 25$ and $f(22) = 487$.

9. If $\frac{4}{5}x = y$ and $x^y = y^x$, then $x - y$ can be expressed as a reduced fraction $\frac{a}{b}$. Find the value of $a + b$.

Solution:

We assume that x and y are positive numbers.
$x^{\frac{4}{5}x} = \left(\frac{4}{5}x\right)^x$ so $\left(x^{\frac{4}{5}}\right)^x = \left(\frac{4}{5}x\right)^x$.
Therefore, $x^{\frac{4}{5}} = \frac{4}{5}x$ and $x = \left(\frac{5}{4}\right)^5$.
Consequently, $y = \frac{4}{5}x = \left(\frac{5}{4}\right)^4$.
$x - y = \left(\frac{5}{4}\right)^5 - \left(\frac{5}{4}\right)^4 = \frac{5^4}{4^5}$ (which is a reduced fraction) so
$a + b = 5^4 + 4^5 = 1649$.

10. Find $(1 + i)^{12345678987654321}$

Solution:

$1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$ and $12345678987654321 \equiv 1 \pmod 8$
Therefore, $(1 + i)^{12345678987654321} = (\sqrt{2})^{12345678987654321}e^{i\frac{\pi}{4}}$

11. Find $\exp\left(\ln(x + 1) + \ln(x^2 + 1) + \ln(x^4 + 1) + \cdots + \ln(x^{128} + 1)\right)$ fully simplified and without any parentheses in the final result.

Solution:

Let's note that number $P(x)$.
Using properties of exp and ln, we can find that, in fact, $P(x) = \prod_{k=0}^{7}(x^{2^k} + 1)$.
Let $x \neq 1$. Then, for all k, $\left(x^{2^k} + 1\right)\left(x^{2^{k+1}} + 1\right) = x^{2^k + 2^{k+1}} + x^{2^k} + x^{2^{k+1}} + 1 = \left(x^{2^k}\right)^3 + \left(x^{2^k}\right)^2 + x^{2^k} + 1 = \frac{\left(x^{2^k}\right)^4 - 1}{x^{2^k} - 1}$ (formula for a geometric sum).
Therefore, $P(x) = \left(\frac{x^4-1}{x-1}\right)\left(\frac{x^{16}-1}{x^4-1}\right)\left(\frac{x^{64}-1}{x^{16}-1}\right)\left(\frac{x^{256}-1}{x^{64}-1}\right)$
There is a lot of simplifications, leaving $P(x) = \frac{x^{256}-1}{x-1}$
Hence, $P(x) = x^{255} + \cdots + x + 1$, which is also true if $x = 1$.

12. Let $F_0 = 0$ and $F_1 = 1$ and for $n \geq 2$ we have $F_{n-2} + F_{n-1} = F_n$. Find $\sum_{k=0}^{\infty} \frac{F_k}{7^k}$.

Solution:

The series converges: we can prove by induction that $F_k \leq 2^k$ for all nonnegative integer k.

So $0 \leq \frac{F_k}{7^k} \leq \left(\frac{2}{7}\right)^k$ and, consequently, the series converge accodring to the comparison test (as $\sum \left(\frac{2}{7}\right)^k$ converges).

Let l be the value of the sum.

Let $u_k = \frac{F_k}{7^k}$ for all nonnegative integer k.
For all integer $k \geq 2$, $u_k = \frac{F_k}{7^k} = \frac{F_{k-1}}{7^k} + \frac{F_{k-2}}{7^k} = \frac{1}{7}u_{k-1} + \frac{1}{49}u_{k-2}$
Let $N \geq 2$ be an integer. $\sum_{k=0}^{N} u_k = u_0 + u_1 + \sum_{k=2}^{N}(\frac{1}{7}u_{k-1} + \frac{1}{49}u_{k-2}) = \frac{1}{7} + \frac{1}{7}\sum_{k=1}^{N-1} u_k + \frac{1}{49}\sum_{k=0}^{N-2} u_k$

Taking the limit ($N \to \infty$) of both sides: $l = \frac{1}{7} + \frac{1}{7}(l - u_0) + \frac{1}{49}l = \frac{8}{49}l + 1$; so $l = \frac{7}{41}$.

13. The sum of the real coefficients of the terms of the expanded form of $(1+i)^{1337}$ can be expressed as $2n$ where n is an integer. Find n.

Solution:

$1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$ and $1337 \equiv 1 \pmod 8$
Therefore, $1 + i = \sqrt{2}^{1337} e^{i\frac{\pi}{4}} = 2^{668}\sqrt{2}(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = 2^{668} + 2^{668}i$ so $n = 668$.

14. Find the solutions to $x^9 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + 1 = 0$.

Solution:

Notice that:
$x^9 + x^7 = x^7(x^2 + 1)$
There is also a $x^2 + 1$, so we can try to group the terms to make this factor appear:
$x^6 + x^4 = x^4(x^2 + 1)$ and
$x^5 + x^3 = x^3(x^2 + 1)$
Therefore, $x^9 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + 1 = (x^2 + 1)(x^7 + x^4 + x^3 + 1)$
You can factorize it further: $x^7 + x^4 + x^3 + 1 = x^4(x^3 + 1) + (x^3 + 1) = (x^4 + 1)(x^3 + 1)$
Hence, this equation is equivalent to "$(x^2 + 1)(x^4 + 1)(x^3 + 1) = 0$"
The only real solution is $x = -1$, but if we include complex numbers, then the solutions are $-1, i, -i, -j, -j^2, e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}$ and $e^{i\frac{7\pi}{4}}$

15. If $a, b, c > 0$, find the minimum possible value of $\frac{a}{2b} + \frac{b}{4c} + \frac{c}{8a}$.

Solution:

According the the AM-GM inequality:

$\frac{a}{2b} + \frac{b}{4c} + \frac{c}{8a} \geq 3\left(\frac{a}{2b} \cdot \frac{b}{4c} \cdot \frac{c}{8a}\right)^{1/3} = 3(\frac{1}{64})^{1/3} = \frac{3}{4}$
which is reached, for example, when $a = 1$, $c = 2$ and $b = 2$.

16. Show that if a polynomial with real coefficients has the root $a + bi$ for $a, b \neq 0$, Then $a - bi$ is also a root.

Solution:

Let P be that polynomial. Let's note $P(X) = a_n X^n + \cdots + a_1 X + a_0$
Let $z = a + bi$. Then:
$P(z) = a_n z^n + \cdots + a_1 z + a_0$
Taking the conjugate:
$0 = \overline{a_n z^n + \cdots + a_1 z + a_0} = \overline{a_n z^n} + \cdots + \overline{a_1 z} + \overline{a_0} = a_n \overline{z}^n + \cdots + a_1 \overline{z} + a_0 = P(\overline{z})$
Therefore, $a - ib$ is also a root of P.

17. For what value of k is the following function continuous at 2? $f(x) = k$ if $x = 2$, $f(x) = \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}$ if $x \neq 2$

Solution:

For all $x \neq 2$:
$f(x) = \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} = \frac{(\sqrt{2x+5}-\sqrt{x+7})(\sqrt{2x+5}+\sqrt{x+7})}{(x-2)(\sqrt{2x+5}+\sqrt{x+7})} = \frac{2x+5-(x+7)}{(x-2)(\sqrt{2x+5}+\sqrt{x+7})} = \frac{1}{\sqrt{2x+5}+\sqrt{x+7}}$
Therefore, $f(x) \to \frac{1}{\sqrt{2 \cdot 2+5}+\sqrt{2+7}} = \frac{1}{6}$ when $x \to 2$, so the function is continuous at 2 iff $k = \frac{1}{6}$

18. Let a, b, c be the roots of $x^3 - x - 3 = 0$. Find $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$.

Solution:

According to Vieta's formulas:
$abc = 3$
$a + b + c = 0$
$ab + bc + ca = -1$

So $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = \frac{(ab+bc+ca)+2(a+b+c)+3}{abc+(ab+bc+ca)+(a+b+c)+1} = \frac{2}{3}$

19. x_1, x_2, x_3,... is an arithmetic sequence. y_1, y_2, y_3,... is a geometric sequence. $z_1, z_2 z_3$,... is a sequence such that for all positive integers n, we have $x_n + y_n = z_n$. If $z_1 = 1, z_2 = 8, z_3 = 10, z_4 = 32$ then find z_5.

Solution:

There exist x, y, p and q such that, for all positive integer n:
$x_n = x + (n-1)p$ and
$y_n = q^{n-1} y$
So $z_n = x + (n-1)p + q^{n-1} y$. The stated equalities give the following system of equations:

$x + y = 1$
$x + p + qy = 8$
$x + 2p + q^2y = 10$
$x + 3p + q^3y = 32$
If we solve it, we find that $x = \frac{6}{5}$, $y = -\frac{1}{5}$, $p = 6$ and $q = -4$.
Therefore, $z_5 = x + 4p + q^4y = -26$.

20. Solve (x + 10)(x + 11)(x + 12)(x + 13) = 1.

Solution:

Let $X = x + \frac{23}{2}$. We have to solve: $(X - \frac{3}{2})(X - \frac{1}{2})(X + \frac{1}{2})(X + \frac{3}{2})$
Using the "$(a - b)(a + b) = a^2 - b^2$" identity:
$(X - \frac{3}{2})(X - \frac{1}{2})(X + \frac{1}{2})(X + \frac{3}{2}) - 1 = (X^2 - \frac{1}{4})(X^2 - \frac{9}{4}) - 1 = X^4 - \frac{5}{2}X^2 - \frac{7}{16}$
Solving the quadratic equation (with $Y = X^2$), we find that $X^2 = \frac{5+4\sqrt{2}}{4}$ (the other solution is rejected because $X^2 \geq 0$).
Therefore, $X = \pm\frac{\sqrt{5+4\sqrt{2}}}{2}$ and $x = \pm\frac{\sqrt{5+4\sqrt{2}}}{2} - \frac{23}{2}$.

21. Find the solution(s) to $\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{a}{3}}}} = 0{,}5$.

Solution:

Suppose there is a solution a.
Then $1 + \frac{1}{1+\frac{1}{1+\frac{a}{3}}} = 2$ so $\frac{1}{1+\frac{1}{1+\frac{a}{3}}} = 1$
Then $1 + \frac{1}{1+\frac{a}{3}} = 1$ so $\frac{1}{1+\frac{a}{3}} = 0$ which is impossible (the inverse of a real number cannot be null).
There is no solution.